



Boundedness in a quasilinear two-species chemotaxis system with consumption of chemoattractant

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Abstract. This paper deals with a two-species chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D_1(u) \nabla u) - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \quad t > 0, \\ v_t = \nabla \cdot (D_2(v) \nabla v) - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - (\alpha u + \beta v)w, & x \in \Omega, \quad t > 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$; χ_i ($i = 1, 2$) are chemotactic functions satisfying $\chi'_i \geq 0$; the parameters $\mu_1, \mu_2 > 0, a_1, a_2 > 0$ and $\alpha, \beta > 0$, the initial data $(u_0, v_0) \in (C^0(\bar{\Omega}))^2$ and $w_0 \in W^{1,\infty}(\Omega)$ are non-negative. Based on the maximal Sobolev regularity, it is shown that this system possesses a unique global bounded classical solution provided that the logistic growth coefficients μ_1 and μ_2 are sufficiently large.

Keywords: two-species chemotaxis system, boundedness, maximal Sobolev regularity.


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1 Introduction

This paper considers the following quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D_1(u) \nabla u) - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \quad t > 0, \\ v_t = \nabla \cdot (D_2(v) \nabla v) - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - (\alpha u + \beta v)w, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$ and ν denotes the outer normal vector to $\partial\Omega$, the constants $\mu_1, \mu_2, a_1, a_2, \alpha$ and β are positive. We consider the

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initial data as follows

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) & \text{with } u_0 \geq 0 \text{ in } \Omega, \\ v_0 \in C^0(\overline{\Omega}) & \text{with } v_0 \geq 0 \text{ in } \Omega, \\ w_0 \in W^{1,\infty}(\Omega) & \text{with } w_0 \geq 0 \text{ in } \Omega. \end{cases} \quad (1.2)$$

The chemotactic sensitivity function $\chi_i(w)$ ($i = 1, 2$) satisfy

$$\chi_i(w) > 0 \quad \text{and} \quad \chi'_i(w) \geq 0. \quad (1.3)$$

Furthermore, we assume that the diffusion function $D_i(s) \in C^2([0, \infty))$ ($i = 1, 2$) as well as

$$D_i(s) \geq c_{D_i}(s+1)^{m-1} \quad \text{for all } s \geq 0, \quad (1.4)$$

where $c_{D_i} > 0$ and $m \in \mathbb{R}$. In model (1.1), $u = u(x, t)$ and $v = v(x, t)$ represent densities of two populations, respectively, and $w = w(x, t)$ denotes the concentration of oxygen.

System (1.1) is used in mathematical biology as a model to study the mechanism of two-species chemotaxis. The model describes the nonlinear diffusion of competing species which move towards the gradient of a substance called chemoattractant. Chemotaxis system plays a crucial role in cellular communication, for instance, in the governing of immune cells migration, in wound healing, in tumours growth or in the organization of embryonic cell positioning (see e.g. [3, 5, 38, 40]).

The classical Keller–Segel model was proposed by Keller and Segel [14], and the existence of traveling wave solutions was proved under some conditions. Based on the Keller–Segel model, various chemotaxis models have attracted many authors to explore their mathematical properties, such as the boundedness, the stabilization of solutions and the blow-up of solutions [4, 6, 8, 12, 17, 18, 21, 23, 24, 27, 34–37, 39, 41].

A typical chemotaxis process is considered where the signal is degraded, but not produced by the cells. More precisely, the following oxygen consumption model is studied

$$\begin{cases} u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \end{cases} \quad (1.5)$$

where u and v represent the density of the bacteria and the concentration of oxygen, respectively. $D(u)$ denotes the diffusion function and $f(u)$ is the logistic source. The analysis of this model has attracted many interests and many results are presented. For instance, in the absence of the logistic source (i.e. $f(u) \equiv 0$), when $D(u) = 1$, the global bounded solutions have been shown by Tao [20] under the condition of $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)\chi}$. For arbitrarily large initial data, in three-dimensional case, the global bounded weak solutions and smoothness in $\Omega \times (T, +\infty)$ are proved with some $T > 0$ by Tao and Winkler [22]. Moreover, when $D(u)$ satisfies (1.4), Wang et al. prove that system (1.5) possesses a unique global bounded classical solution if $m > \frac{1}{2}$ in the case $n = 1$ or $m > 2 - \frac{2}{n}$ in the case $n \geq 2$ [32], the domain can be extended to $m > 2 - \frac{6}{n+4}$ in the case $n \geq 3$, but the solutions maybe unbounded in [31]. Furthermore, the global bounded solutions are proved [9, 33] provided that $m > 2 - \frac{n+2}{2n}$, which improves the results in [31, 32]. Recently, the diffusivity $D(u)$ exponential decay as $u \rightarrow \infty$ is studied in [16, 26].

If the logistic source $f(u) = au - \mu u^\gamma$ with $\gamma > 1$ and $D(u) = \delta$ in system (1.5), the global bounded solution is studied if

$$\|v_0\|_{L^\infty(\Omega)} < \frac{1}{\chi} \sqrt{\frac{\delta}{2(n+1)}} \left[\pi - 2 \arctan \frac{\delta-1}{2} \sqrt{\frac{2(n+1)}{\delta}} \right]$$

in [2]. Similarly, Lankeit and Wang [15] prove this system has global bounded solutions if $\mu > c_1(n)\|\chi v_0\|_{L^\infty(\Omega)}^{\frac{1}{n}} + c_2(n)\|\chi v_0\|_{L^\infty(\Omega)}^{2n}$, where $c_1(n)$ and $c_2(n)$ are constants about n . The chemotaxis-consumption model (1.5) with nonlinear diffusion function and nontrivial source terms has also already been considered in [28, 30].

To better discuss model (1.1), we need to mention the following two species chemotaxis(-Navier)-Stokes system with Lotka-Volterra competitive kinetics [25]

$$\begin{cases} u_t + V \cdot \nabla u = \Delta u - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \quad t > 0, \\ v_t + V \cdot \nabla v = \Delta v - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \quad t > 0, \\ w_t + V \cdot \nabla w = \Delta w - (\alpha u + \beta v)w, & x \in \Omega, \quad t > 0, \\ V_t + \kappa(V \cdot \nabla V) = \Delta V - \nabla P + (\gamma u + \delta v)\nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot V = 0, & x \in \Omega, \quad t > 0, \end{cases} \quad (1.6)$$

which describes the evolution of two competing species that reacts on a chemoattractant in the environment of fulling the fluid. Here u, v and w are represented as model (1.1), and V denotes the velocity field of the fluid belonging to an incompressible Navier-Stokes equation with pressure P . Moreover, ϕ is a potential function, and κ is a constant concerning the strength of nonlinear fluid convection. Boundedness and asymptotic behavior of model (1.6) are researched in the case two-dimension and three-dimension [7, 11, 13]. When the fluid is stationary or the effect of fluid is absent, i.e. $V \equiv 0$, model (1.6) is ascribed to the fundamental chemotaxis model (1.1).

Motivated by the arguments in [19, 29, 30, 37, 41], in this paper, we extend their method and then obtain global boundedness of solution of model (1.1). Our main results are as follows.

Theorem 1.1. *Assume $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary, $\chi_i(w)$ ($i = 1, 2$) satisfy (1.3), and $D_1(u)$ and $D_2(v)$ satisfy (1.4). Moreover, assume that there exists $\mu_0 > 0$ such that $\min\{\mu_1, \mu_2\} > \mu_0$. Then for the initial data (u_0, v_0, w_0) satisfies (1.2), system (1.1) possesses a unique classical solution (u, v, w) which is uniformly bounded in the sense that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} < C \quad \text{for all } t > 0 \quad (1.7)$$

with some constants $C > 0$.

Remark 1.2. For $i = 1, 2$, when $D_i(s) = d_i > 0$ is constant, if

$$0 < \|w_0\|_{L^\infty(\Omega)} \leq \frac{1}{3(n+1)\|\chi_i\|_{L^\infty[0, \|w_0\|_{L^\infty(\Omega)}]}} \min\left\{\frac{2d_i}{d_i+1}, 1\right\},$$

model (1.1) has global bounded solutions in [29], but which is independent of μ_1 and μ_2 . Theorem 1.1 gives a qualitative result, namely, if μ_i ($i = 1, 2$) are sufficiently large, model (1.1) has global bounded solutions, which improves above results in some sense.

The rest of this paper is organized as follows. In the next section, we show the local existence of a solution to model (1.1) and give some preliminary inequalities those are important for our proofs. In Section 3, we will give the complete proof of Theorem 1.1.

2 Preliminaries

In order to prove our result, we first give one result concerning local-in-time existence of a classical solution to system (1.1).

Lemma 2.1. Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary, $\mu_1, \mu_2 > 0, \alpha, \beta > 0$ and $a_1, a_2 > 0$. Moreover, assume that the initial data (u_0, v_0, w_0) satisfies (1.2), $\chi_i(w)$ ($i = 1, 2$) satisfy (1.3), and $D_1(u)$ and $D_2(v)$ satisfy (1.4). Then there exists $t \in (0, T_{\max})$ such that system (1.1) has a unique local-in-time non-negative triple solution

$$u, v, w \in C(\overline{\Omega} \times (0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})). \quad (2.1)$$

In addition, if $T_{\max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \rightarrow \infty \quad \text{as } t \nearrow T_{\max}. \quad (2.2)$$

Proof. Let $U = (u, v, w) \in \mathbb{R}^n (n \geq 1)$. And (1.1) can be transformed to

$$\begin{cases} U_t = \nabla \cdot (A(U) \nabla U) + F(U), \\ \frac{\partial U}{\partial \nu} = 0, \\ U(x, 0) = (u_0(x), v_0(x), w_0(x)), \end{cases} \quad \begin{matrix} x \in \partial\Omega, \quad t > 0, \\ x \in \Omega, \end{matrix} \quad (2.3)$$

where

$$A(U) = \begin{pmatrix} D_1(u) & 0 & -\chi_1(w) \\ 0 & D_2(v) & -\chi_2(w) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F(U) = \begin{pmatrix} \mu_1 u(1 - u - a_1 v) \\ \mu_2 v(1 - a_2 u - v) \\ -(\alpha u + \beta v)w \end{pmatrix}.$$

Since the eigenvalues of A are positive, system (2.3) is normally parabolic. Applying Theorems 14.4, 14.6 and 15.5 of [1], (2.1) and (2.2) can be proved. And the initial data satisfies (1.2), the maximum principle ensures that u, v and w are non-negative in $\Omega \times (0, T_{\max})$. \square

The following characteristic of the solution of the third equation in model (1.1) plays an essential role in the later proof.

Lemma 2.2. Let (u, v, w) be the solution of model (1.1), then we have

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} \quad (2.4)$$

for all $t \in (0, T_{\max})$.

Proof. According to the third equation of model (1.1), and the non-negative u, v, w and $\alpha, \beta > 0$, we claim result (2.4) upon an application of the maximum principle. \square

Finally, we provide the result referred to as a variation of Maximal Sobolev regularity, which was proposed in Theorem 3.1 in [10] (see also Lemma 3.1 in [6], Lemma 2.2 in [37] and Lemma 2.2 in [30]).

Lemma 2.3. Assume that $T \in (0, \infty)$, we mention the following homogeneous heat equations

$$\begin{cases} y_t = \Delta y - f y, & x \in \Omega, \quad t \in (0, T), \\ \frac{\partial y}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T), \\ y(x, 0) = y_0(x), & x \in \Omega, \end{cases} \quad (2.5)$$

where $y_0 \in W^{2,\theta}(\Omega)$ ($\theta > 1$) is non-negative with $\frac{\partial y_0}{\partial \nu} = 0$ on $\partial\Omega$ and any functions $f \in L^\theta((0, T); L^\theta(\Omega))$ are non-negative, there exists a unique solution

$$y \in W^{1,\theta}((0, T); L^\theta(\Omega)) \cap L^\theta((0, T); W^{2,\theta}(\Omega)),$$

and

$$\begin{aligned} & \int_0^T \int_{\Omega} y^\theta dxdt + \int_0^T \int_{\Omega} |y_t|^\theta dxdt + \int_0^T \int_{\Omega} |\Delta y|^\theta dxdt \\ & \leq C_\theta \left(\int_0^T \int_{\Omega} (fy)^\theta dxdt + \int_{\Omega} y_0^\theta dx + \int_{\Omega} |\Delta y_0|^\theta dx \right), \end{aligned} \quad (2.6)$$

with some constant $C_\theta > 0$. Moreover, for $s \in (0, T)$, $y(\cdot, s) \in W^{2,\theta}(\Omega)$ ($\theta > 1$) with $\frac{\partial y(\cdot, s)}{\partial \nu} = 0$ on $\partial\Omega$, then

$$\begin{aligned} & \int_s^T \int_{\Omega} y^\theta dxdt + \int_s^T \int_{\Omega} |y_t|^\theta dxdt + \int_s^T \int_{\Omega} |\Delta y|^\theta dxdt \\ & \leq C_\theta \left(\int_s^T \int_{\Omega} (fy)^\theta dxdt + \int_{\Omega} y^\theta(\cdot, s) dx + \int_{\Omega} |\Delta y(\cdot, s)|^\theta dx \right), \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} & \int_s^T \int_{\Omega} e^{\theta t} |\Delta y|^\theta dxdt \leq C_\theta \int_s^T \int_{\Omega} e^{\theta t} y^\theta |1 - f|^\theta dxdt \\ & \quad + C_\theta \int_{\Omega} y^\theta(\cdot, s) dx + C_\theta \int_{\Omega} |\Delta y(\cdot, s)|^\theta dx. \end{aligned} \quad (2.8)$$

Proof. (2.6) and (2.7) are proved in [6]. Now we prove (2.8). Similar to Lemma 2.2 in [37], let $z(x, \tau) = e^\tau y(x, \tau)$, then we have

$$\begin{cases} z_\tau = \Delta z + e^\tau y(1 - f), & \Omega \times (0, T), \\ \frac{\partial z}{\partial \nu} = 0, & \partial\Omega \times (0, T), \\ z(x, 0) = z_0(x), & x \in \Omega. \end{cases}$$

Using Theorem 3.1 in [10], we get

$$\int_0^T \int_{\Omega} |\Delta z|^\theta dx d\tau \leq C_0 \left(\int_0^T \int_{\Omega} e^{\theta \tau} y^\theta |1 - f|^\theta dx d\tau + \int_{\Omega} y_0^\theta dx + \int_{\Omega} |\Delta y_0|^\theta dx \right),$$

which implies

$$\int_0^T \int_{\Omega} e^{\theta \tau} |\Delta y|^\theta dx d\tau \leq C_0 \left(\int_0^T \int_{\Omega} e^{\theta \tau} y^\theta |1 - f|^\theta dx d\tau + \int_{\Omega} y_0^\theta dx + \int_{\Omega} |\Delta y_0|^\theta dx \right).$$

Hence, we replace $y(\tau)$ by $y(\tau + s)$. Then, the inequality (2.8) is obtained. \square

3 Global boundedness

In this section, global boundedness of solutions is proved to model (1.1). Firstly, to prove Theorem 1.1, we make an estimate for $(u, v, w, \Delta w)$ when $s_0 \in (0, T_{\max})$ and $s_0 < 1$. According to Lemma 2.1, it shows that $u(\cdot, s_0), v(\cdot, s_0), w(\cdot, s_0) \in C^2(\overline{\Omega})$ with $\frac{\partial w(\cdot, s_0)}{\partial \nu} = 0$ on $\partial\Omega$. Subsequently, we pick $M_0 > 0$ such that

$$\begin{cases} \sup_{0 \leq t \leq s_0} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq M_0, & \sup_{0 \leq t \leq s_0} \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq M_0, \\ \sup_{0 \leq t \leq s_0} \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq M_0, & \|\Delta w(\cdot, t)\|_{L^\infty(\Omega)} \leq M_0. \end{cases} \quad (3.1)$$

Next, we prove boundedness in $t \in (s_0, T_{\max})$.

Lemma 3.1. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with smooth boundary and $\chi_i(w)$ ($i = 1, 2$) satisfy (1.3). For any $p > 1$ and $\eta > 0$, there exists $\mu_{p,\eta} > 0$ such that if $\min\{\mu_1, \mu_2\} > \mu_{p,\eta}$, then

$$\|u(\cdot, t)\|_{L^p(\Omega)} + \|v(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (s_0, T_{\max}) \quad (3.2)$$

where $C = C(p, |\Omega|, \mu_1, \mu_2, \eta, u_0, v_0, w_0) > 0$.

Proof. By direct calculations, we obtain from the first and third equations in model (1.1) that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p &= \int_{\Omega} u^{p-1} [\nabla \cdot (D_1(u) \nabla u) - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u (1 - u - a_1 v)] \\ &= - \int_{\Omega} (p-1) u^{p-2} D_1(u) |\nabla u|^2 + \int_{\Omega} (p-1) u^{p-1} \chi_1(w) \nabla u \cdot \nabla w \\ &\quad + \mu_1 \int_{\Omega} u^p - \mu_1 \int_{\Omega} u^{p+1} - \mu_1 a_1 \int_{\Omega} u^p v \\ &\leq \frac{p-1}{p} \int_{\Omega} \nabla u^p \cdot \nabla \Phi_1(w) + \mu_1 \int_{\Omega} u^p - \mu_1 \int_{\Omega} u^{p+1} \\ &= - \frac{p-1}{p} \int_{\Omega} u^p \chi_1(w) \Delta w - \frac{p-1}{p} \int_{\Omega} u^p \chi'_1(w) |\nabla w|^2 + \mu_1 \int_{\Omega} u^p - \mu_1 \int_{\Omega} u^{p+1}, \end{aligned}$$

where $\Phi_i(w) = \int_1^w \chi_i(s) ds$ ($i = 1, 2$), so we have $\nabla \Phi_i(w) = \chi_i(w) \nabla w$ and $\Delta \Phi_i(w) = \chi'_i(w) |\nabla w|^2 + \chi_i(w) \Delta w$. Thanks to $\chi'_i(w) \geq 0$ ($i = 1, 2$), we arrive at

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq - \frac{p+1}{p} \int_{\Omega} u^p - \frac{p-1}{p} \int_{\Omega} u^p \chi_1(w) \Delta w + \left(\mu_1 + \frac{p+1}{p} \right) \int_{\Omega} u^p - \mu_1 \int_{\Omega} u^{p+1} \quad (3.3)$$

for all $t \in (s_0, T_{\max})$. For any $\varepsilon > 0$, based on Young's inequality, we conclude

$$\left(\mu_1 + \frac{p+1}{p} \right) \int_{\Omega} u^p \leq \varepsilon \int_{\Omega} u^{p+1} + c_1 |\Omega| \quad (3.4)$$

and

$$\begin{aligned} - \frac{p-1}{p} \int_{\Omega} u^p \chi_1(w) \Delta w &\leq \int_{\Omega} u^p \chi_1(w) |\Delta w| \leq M_1 \int_{\Omega} u^p |\Delta w| \\ &\leq \eta \int_{\Omega} u^{p+1} + c_2 \eta^{-p} M_1^{p+1} \int_{\Omega} |\Delta w|^{p+1}, \end{aligned} \quad (3.5)$$

where $\chi_i(w) \leq M_i := \chi_i(\|w_0\|_{L^\infty(\Omega)})$ due to $\chi'_i(w) \geq 0$ ($i = 1, 2$) and (2.4), and constants $c_1 = \frac{1}{p+1} (1 + \frac{1}{p})^{-p} \varepsilon^{-p} (\mu_1 + \frac{p+1}{p})^{p+1} > 0$ and $c_2 = \sup_{p>1} \frac{1}{p+1} (1 + \frac{1}{p})^{-p} < \infty$. Inserting (3.4) and (3.5) into (3.3), we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{p} \int_{\Omega} u^p \right) &\leq - (p+1) \left(\frac{1}{p} \int_{\Omega} u^p \right) - (\mu_1 - \varepsilon - \eta) \int_{\Omega} u^{p+1} \\ &\quad + c_2 \eta^{-p} M_1^{p+1} \int_{\Omega} |\Delta w|^{p+1} + c_1 |\Omega|. \end{aligned} \quad (3.6)$$

Applying the variation-of-constants formula to the inequality (3.6), it shows that

$$\begin{aligned} \frac{1}{p} \int_{\Omega} u^p(\cdot, t) &\leq e^{-(p+1)(t-s_0)} \frac{1}{p} \int_{\Omega} u^p(\cdot, s_0) - (\mu_1 - \varepsilon - \eta) \int_{s_0}^t e^{-(p+1)(t-s)} \int_{\Omega} u^{p+1} \\ &\quad + c_2 \eta^{-p} M_1^{p+1} \int_{s_0}^t e^{-(p+1)(t-s)} \int_{\Omega} |\Delta w|^{p+1} + c_1 |\Omega| \int_{s_0}^t e^{-(p+1)(t-s)} \\ &\leq - (\mu_1 - \varepsilon - \eta) e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} u^{p+1} \\ &\quad + c_2 \eta^{-p} M_1^{p+1} e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} |\Delta w|^{p+1} + c_3 \end{aligned} \quad (3.7)$$

for all $t \in (s_0, T_{\max})$, where $c_3 = c_1|\Omega|\frac{1}{p+1} + \frac{1}{p} \int_{\Omega} u^p(\cdot, s_0) > 0$. According to Lemma 2.3, there exists $C_p > 0$ such that

$$\begin{aligned} & \int_{s_0}^t \int_{\Omega} e^{(p+1)s} |\Delta w|^{p+1} \\ & \leq C_p \int_{s_0}^t \int_{\Omega} e^{(p+1)s} w^{p+1} |1 - (\alpha u + \beta v)|^{p+1} + C_p \int_{\Omega} w^{p+1}(\cdot, s_0) + C_p \int_{\Omega} |\Delta w(\cdot, s_0)|^{p+1} \\ & \leq C_p \int_{s_0}^t \int_{\Omega} e^{(p+1)s} w^{p+1} |(\alpha u + \beta v) + 1|^{p+1} + C_p \int_{\Omega} w^{p+1}(\cdot, s_0) + C_p \int_{\Omega} |\Delta w(\cdot, s_0)|^{p+1}. \end{aligned}$$

Thanks to the inequality $(a + b)^d \leq 2^d(a^d + b^d)$ with $a, b \geq 0$ and $d \geq 1$, we have

$$\begin{aligned} & c_2 \eta^{-p} M_1^{p+1} e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} |\Delta w|^{p+1} \\ & \leq c_2 \eta^{-p} M_1^{p+1} C_p e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} w^{p+1} 2^{p+1} [1 + (\alpha u + \beta v)^{p+1}] \\ & \quad + c_2 \eta^{-p} M_1^{p+1} C_p e^{-(p+1)t} \left(\int_{\Omega} w^{p+1}(\cdot, s_0) + \int_{\Omega} |\Delta w(\cdot, s_0)|^{p+1} \right) \\ & \leq c_2 \eta^{-p} M_1^{p+1} C_p \int_{s_0}^t \int_{\Omega} e^{-(p+1)(t-s)} w^{p+1} [2^{p+1} + 2^{2p+2}(\alpha u)^{p+1} + 2^{2p+2}(\beta v)^{p+1}] \\ & \quad + c_2 \eta^{-p} M_1^{p+1} C_p e^{-(p+1)t} \|w(\cdot, s_0)\|_{W^{2,p+1}(\Omega)}^{p+1} \\ & = c_2 \eta^{-p} M_1^{p+1} e^{-(p+1)t} \left[c_4 \int_{s_0}^t \int_{\Omega} e^{(p+1)s} u^{p+1} + c_5 \int_{s_0}^t \int_{\Omega} e^{(p+1)s} v^{p+1} \right] \\ & \quad + c_2 \eta^{-p} M_1^{p+1} c(t), \end{aligned} \tag{3.8}$$

where $c_4 = C_p 2^{2p+2} \alpha^{p+1} \|w_0\|_{L^\infty(\Omega)}^{p+1}$, $c_5 = C_p 2^{2p+2} \beta^{p+1} \|w_0\|_{L^\infty(\Omega)}^{p+1}$, and

$$\begin{aligned} c(t) &= C_p e^{-(p+1)t} \|w(\cdot, s_0)\|_{W^{2,p+1}(\Omega)}^{p+1} + C_p 2^{p+1} \int_{s_0}^t \int_{\Omega} e^{-(p+1)(t-s)} w^{p+1} \\ &\leq C_p \|w(\cdot, s_0)\|_{W^{2,p+1}(\Omega)}^{p+1} + C_p 2^{p+1} \|w_0\|_{L^\infty(\Omega)}^{p+1} |\Omega| \int_{s_0}^t e^{-(p+1)(t-s)} ds \\ &= C_p \|w(\cdot, s_0)\|_{W^{2,p+1}(\Omega)}^{p+1} + \frac{|\Omega|}{p+1} C_p 2^{p+1} \|w_0\|_{L^\infty(\Omega)}^{p+1} =: c_6. \end{aligned}$$

Inserting (3.8) into (3.7), we obtain

$$\begin{aligned} \frac{1}{p} \int_{\Omega} u^p(\cdot, t) &\leq -(\mu_1 - \varepsilon - \eta) e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} u^{p+1} \\ &\quad + c_2 c_4 \eta^{-p} M_1^{p+1} e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} u^{p+1} \\ &\quad + c_2 c_5 \eta^{-p} M_1^{p+1} e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} v^{p+1} + c_7 \end{aligned} \tag{3.9}$$

with some $c_7 > 0$. Similarly,

$$\begin{aligned} \frac{1}{p} \int_{\Omega} v^p(\cdot, t) &\leq -(\mu_2 - \varepsilon - \eta) e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} v^{p+1} \\ &\quad + c_2 c_5 \eta^{-p} M_2^{p+1} e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} v^{p+1} \\ &\quad + c_2 c_4 \eta^{-p} M_2^{p+1} e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} u^{p+1} + c_8 \end{aligned} \tag{3.10}$$

with some $c_8 > 0$. Adding (3.9) and (3.10), we have

$$\begin{aligned} & \frac{1}{p} \left[\int_{\Omega} u^p(\cdot, t) + \int_{\Omega} v^p(\cdot, t) \right] \\ & \leq -(\mu_1 - \varepsilon - \eta - c_2 c_4 \eta^{-p} M_1^{p+1} - c_2 c_4 \eta^{-p} M_2^{p+1}) e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} u^{p+1} \\ & \quad - (\mu_2 - \varepsilon - \eta - c_2 c_5 \eta^{-p} M_2^{p+1} - c_2 c_5 \eta^{-p} M_1^{p+1}) e^{-(p+1)t} \int_{s_0}^t \int_{\Omega} e^{(p+1)s} v^{p+1} + c_9 \end{aligned} \quad (3.11)$$

with some $c_9 > 0$.

Let $\mu_{p,\eta} = \max \{ \eta + c_2 c_4 \eta^{-p} M_1^{p+1} + c_2 c_4 \eta^{-p} M_2^{p+1}, \eta + c_2 c_5 \eta^{-p} M_2^{p+1} + c_2 c_5 \eta^{-p} M_1^{p+1} \}$, we can choose $\varepsilon \in (0, \min\{\mu_1, \mu_2\} - \mu_{p,\eta})$ such that

$$\mu_1 - \varepsilon - \eta - c_2 c_4 \eta^{-p} M_1^{p+1} - c_2 c_4 \eta^{-p} M_2^{p+1} > 0$$

and

$$\mu_2 - \varepsilon - \eta - c_2 c_5 \eta^{-p} M_2^{p+1} - c_2 c_5 \eta^{-p} M_1^{p+1} > 0.$$

Hence, using (3.11), we conclude

$$\frac{1}{p} \left[\int_{\Omega} u^p(\cdot, t) + \int_{\Omega} v^p(\cdot, t) \right] \leq c_9$$

for all $t \in (s_0, T_{\max})$, with some constant $c_9 = c_9(\mu_1, \mu_2, \varepsilon, \eta, p, w(s_0))$. \square

Now our main result can be easily obtained.

Proof of Theorem 1.1. Applying Moser-type iteration techniques, which can be found in Lemma A.1 in [21] (see also [10]). Firstly, we claim that there is a constant $p_0 > n$, such that if

$$\|u(\cdot, t)\|_{L^p(\Omega)} + \|v(\cdot, t)\|_{L^p(\Omega)} < \infty$$

for all $p \geq p_0$ and $t \in (s_0, T_{\max})$, then there exists $c_{10} > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_{10} \quad (3.12)$$

for all $t \in (s_0, T_{\max})$. Assume that μ_0 satisfies

$$\inf_{\eta > 0} \mu_{p_0, \eta} = \inf_{\eta > 0} \left(\max \left\{ \eta + c_2 c'_4 \eta^{-p_0} (M_1^{p_0+1} + M_2^{p_0+1}), \eta + c_2 c'_5 \eta^{-p_0} (M_2^{p_0+1} + M_1^{p_0+1}) \right\} \right) = \mu_0,$$

where $c'_4 = C_{p_0} 2^{2p_0+2} \alpha^{p_0+1} \|w_0\|_{L^\infty(\Omega)}^{p_0+1}$ and $c'_5 = C_{p_0} 2^{2p_0+2} \beta^{p_0+1} \|w_0\|_{L^\infty(\Omega)}^{p_0+1}$. According to $\min\{\mu_1, \mu_2\} > \mu_0$, we have $\min\{\mu_1, \mu_2\} > \mu_{p_0, \eta}$ for some $\eta > 0$. Hence, using Lemma 3.1 implies (3.12) is true for $t \in (s_0, T_{\max})$. Due to (3.1) and Lemma 2.2 we obtain that u, v, w are bounded in $(0, T_{\max})$. Finally, in view of Lemma 2.1 we can complete the proof of Theorem 1.1. \square

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